Mid-Semester Exam ISI Bangalore September 2018

Compact Riemann Surfaces 100 Points

Notes.
(a) There are a total of 105 points in this paper. You will awarded at most 100.
(b) Justify all your steps. Use only those results that have been proved in class unless you have been asked to prove the same.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) $\mathcal{O}(-)$ = the ring of holomorphic functions on a given Riemann surface, while $\mathcal{M}(-)$ = the field of meromorphic functions on it.
(e) $\mathbb{P}^1$ = the Riemann sphere.

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1. [15 points] Define the notion of a meromorphic function on a Riemann surface X. Prove that there is a 1-1 correspondence between nonconstant meromorphic functions on X and nonconstant holomorphic maps from X to $\mathbb{P}^1$.

2. [15 points] Classify the automorphisms of $\mathbb{P}^1$. More precisely, prove that biholomorphic maps from $\mathbb{P}^1$ to itself are in 1-1 correspondence with fractional linear transformations of the complex plane, i.e., maps given by

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0.$$

3. [15 points]
   (i) Let $\Gamma, \Gamma' \subset \mathbb{C}$ be two lattices. Suppose $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$. Show that the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto \alpha z$ induces a holomorphic map $\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ which is a biholomorphism if and only if $\alpha\Gamma = \Gamma'$.
   (ii) Show that every torus $X = \mathbb{C}/\Gamma$ is isomorphic to a torus of the form $X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\operatorname{Im}(\tau) > 0$.

4. [20 points]
   (i) Give an example of a nonconstant entire function $f\colon \mathbb{C} \to \mathbb{C}$ which is not proper.
   (ii) Prove that a polynomial map $p(z)\colon \mathbb{C} \to \mathbb{C}$ is proper.
   (iii) Prove that the branch points of a polynomial map $p(z)\colon \mathbb{C} \to \mathbb{C}$ are precisely those where $p'(z) = 0$.

5. [20 points] Consider the polynomial map $p\colon \mathbb{C} \to \mathbb{C}$ given by $p(z) = z^3 - z$.
   (i) Find the largest open subsets $Y, X \subset \mathbb{C}$ for which the restriction $p|_Y \colon Y \to X$ is an unbranched covering.
   (ii) For X, Y as in (i), calculate $\operatorname{Deck}(Y/X)$ and determine if $p|_Y$ is Galois.

[Note: You may use the results from the other questions even if you haven't proved them.]

6. [20 points] Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X and let $p\colon |\mathcal{F}| \to X$ be the associated total space of $\mathcal{F}$. For any open $U \subset X$, let $\widetilde{\mathcal{F}}(U)$ denote the space of all continuous sections of p over U, i.e., the space of all continuous maps $f\colon U \to |\mathcal{F}|$ with $p \circ f = 1_U$. Let $\widetilde{\mathcal{F}}$ denote the presheaf resulting from the obvious natural restriction maps. Prove the following.

(i) $\widetilde{\mathcal{F}}$ is a sheaf.

(ii) For every open $U \subset X$, there is a natural map $\mathcal{F}(U) \to \widetilde{\mathcal{F}}(U)$ which is an isomorphism if $\mathcal{F}$ is a sheaf.

(iii) There is a natural isomorphism of stalks $\mathcal{F}_x \xrightarrow{\sim} \widetilde{\mathcal{F}}_x$ for every $x \in X$.